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Spectral Concentration near Embedded Eigenvalues

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Techniques of Rayleigh–Schrödinger perturbation theory usually employed for perturbation of isolated eigenvalues are used to obtain theorems on spectral concentration near eigenvalues which are not assumed to be isolated. If $\{H_\kappa\}$ is a family of self adjoint operators convergent in the strong resolvent sense to a self adjoint operator H_0 and λ_0 is an eigenvalue of finite multiplicity of H_0, then the spectrum of $\{H_\kappa\}$ is concentrated near λ_0. Moreover, conditions under which concentration still occurs near λ_0 without the assumption of finite multiplicity are obtained in the semibounded case. © 1990 Academic Press, Inc.

1. INTRODUCTION

Let H_0 be a self adjoint operator in a complex Hilbert space $\mathscr{H}$ with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. Let λ_0 be an eigenvalue of H_0 with corresponding eigenprojection P_0. Further let $\{H_\kappa\}$, $0<\kappa\leqslant\kappa_0$, be a family of self adjoint operators in $\mathscr{H}$ with $H_\kappa\to H_0$ in the strong resolvent sense as $\kappa\to 0^+$.

If λ_0 is isolated with finite multiplicity m and isolation distance $2d$, λ_0 is stable with respect to the family $\{H_\kappa\}$ if for κ sufficiently small $(\lambda_0-d,\lambda_0+d)\cap\sigma(H_\kappa)$ consists of exactly m eigenvalues of H_κ counted according to multiplicity, each of which converges to λ_0 as $\kappa\to 0^+$. Herein, $\sigma(H_\kappa)$ denotes the spectrum of H_κ. This implies uniform convergence of the corresponding family of total projections of H_κ; i.e., the projections onto the m-dimensional space of eigenvectors of H_κ corresponding to the eigenvalues which converge to λ_0, to P_0.

We consider perturbations $\{H_\kappa\}$ of H_0 for which λ_0 is not (necessarily) stable. In particular, we explore the phenomenon of spectral concentration near nonisolated eigenvalues of H_0 as arises, e.g., in the mathematical

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theory associated with Auger or autoionizing states in helium (cf. [8, 10]). Previous results on spectral concentration near embedded eigenvalues (cf. [5, 10]) have been obtained under fairly strong conditions such as dilatation analyticity or other analytic continuation hypotheses. Our method is patterned after techniques traditionally used for perturbation of isolated eigenvalues (cf. [1, 6, 8, 9]). This yields the general result that there is spectral concentration whenever λ_0 has finite multiplicity and H_κ converges to H_0 in the strong resolvent sense, even if λ_0 is a threshold eigenvalue ([10]), or H_0 has dense point spectrum. Moreover, we show that spectral concentration often occurs near eigenvalues of infinite multiplicity. Thus spectral concentration itself may not be of as much physical interest as the complex resonance energies of earlier theories.

2. Eigenvalues of Finite Multiplicity

Throughout this section we assume that λ_0 is an eigenvalue of H_0 of finite multiplicity m with corresponding eigenprojection P_0, and that $\phi_1, ..., \phi_m$ is an orthonormal basis for the λ_0-eigenspace, $P_0\mathscr{H}$. Let $\{\mathbf{I}_\kappa\}$, $0<\kappa\leqslant\kappa_0$, be a family of Borel subsets of $\mathbf{R}$, the real line. Denote by $E_\kappa(\cdot)$, $0\leqslant\kappa\leqslant\kappa_0$, the right continuous resolution of the identity for H_κ, and for a Borel set $\mathbf{I}\subset\mathbf{R}$, let $E_\kappa[\mathbf{I}]$ be the corresponding spectral measure of $\mathbf{I}$. Then [8] *the part of the spectrum of H_κ in $\mathbf{I}_\kappa$ is asymptotically the part of the spectrum of H_0 in $\mathbf{I}$* if $E_\kappa[\mathbf{I}_\kappa]\rightarrow_s E_0[\mathbf{I}]$ as $\kappa\rightarrow 0^+$, where $\rightarrow_s$ denotes strong convergence. If, in addition, $\mathbf{I}=\{\lambda_0\}$ and the Lebesque measure of $\mathbf{I}_\kappa$, $|\mathbf{I}_\kappa|$ converges to zero as $\kappa\rightarrow 0^+$, the spectrum of $\{H_\kappa\}$ is said to be *concentrated near* λ_0. If moreover, $|\mathbf{I}_\kappa|=\mathbf{o}(\kappa^p)$ as $\kappa\rightarrow 0^+$, $p\geqslant 0$, the spectrum of $\{H_\kappa\}$ is *concentrated to order p near* λ_0. $\mathcal{O}$, $\mathbf{o}$ are the usual Landau symbols, and the limiting process is always as $\kappa\rightarrow 0^+$, unless stated otherwise.

Note that when λ_0 is isolated, $\mathbf{I}$ is an isolating interval for λ_0, and $\mathbf{I}_\kappa\subset\mathbf{I}$, the preceding definition of concentration near λ_0 is equivalent to $E_\kappa[\mathbf{I}\backslash\mathbf{I}_\kappa]\rightarrow_s 0$ as $\kappa\rightarrow 0^+$. This is the standard definition of spectral concentration near an isolated eigenvalue [6, 8, 9]. To study spectral concentration near (possibly) embedded eigenvalues by perturbation theoretic methods usually used for isolated point spectra, the preceding terminology needs to be supplemented as follows. Suppose we can find m pairs $(\lambda_{1,\kappa}, \phi_{1,\kappa}), ..., (\lambda_{m,\kappa}, \phi_{m,\kappa})$ in $\mathbf{R}\times\mathscr{D}(H_\kappa)$ such that for each $j=1, ..., m$, $\|(H_\kappa-\lambda_{j,\kappa})\,\phi_{j,\kappa}\|=\mathbf{o}(\kappa^p)$, $\phi_{j,\kappa}\rightarrow\phi_j$, and $\lambda_{j,\kappa}\rightarrow\lambda_j$ as $\kappa\rightarrow 0^+$, where $\mathscr{D}(H_\kappa)$ is the domain of H_κ. Then $\{(\lambda_{j,\kappa}, \phi_{j,\kappa})\colon j=1, ..., m\}$ is called a *set of p-pairs for $\{H_\kappa\}$ near λ_0*. $\lambda_{j,\kappa}$ is called a *pseudoeigenvalue* with corresponding *pseudoeigenvector* $\phi_{j,\kappa}$ and $\{\phi_{1,\kappa}, ..., \phi_{m,\kappa}\}$ is called an *asymptotic basis of order p for $E_\kappa[\cdot]$ near λ_0*.

THEOREM 2.1. *Assume that* $H_\kappa \to H_0$ *in the strong resolvent sense as* $\kappa \to 0^+$. *Then*:

(i) *if* λ_0 *has multiplicity* $m < \infty$ *and there exists a set of p-pairs for* $\{H_\kappa\}$ *near* λ_0, *the spectrum of* $\{H_\kappa\}$ *is concentrated to order p near* λ_0, *and* $\mathbf{I}_\kappa$ *can be taken as the union of m intervals of length* $\mathbf{o}(\kappa^p)$ *centered on the pseudoeigenvalues*, $\lambda_{j,\kappa}$;

(ii) *if* λ_0 *is simple and the spectrum of* $\{H_\kappa\}$ *is concentrated to order p near* λ_0 *with* $\mathbf{I}_\kappa$ *an interval of length* $\mathbf{o}(\kappa^p)$ *whose center* λ_κ *converges to* λ_0 *as* $\kappa \to 0^+$, *then there exists a p-pair for* $\{H_\kappa\}$ *near* λ_0.

The following lemma is basic to the proof of part (i) of Theorem 2.1.

LEMMA 2.2. *If* $H_\kappa \to H_0$ *in the strong resolvent sense*, $\{\mu_\kappa\}$, $0 < \kappa \leqslant \kappa_0$, *is any family of real numbers converging to* λ *as* $\kappa \to 0^+$, *and* $P = E_0(\lambda) - E_0(\lambda - 0)$, *then*

$$E_\kappa(\mu_\kappa)(1-P) \to_s E_0(\lambda - 0) \qquad as \quad \kappa \to 0^+.$$

The lemma reduces to a well-known theorem of self adjoint operator theory when λ is not an eigenvalue of H_0, and is proved by simply following the proof of [6, Theorem VIII.1.15].

Proof of Theorem 2.1. Given Lemma 2.2, the proof closely follows that of the corresponding theorem for isolated eigenvalues [6, 8, 9]. Let $\varepsilon_j(\kappa) = \|(H_\kappa - \lambda_{j,\kappa})\,\phi_{j,\kappa}\|$, and pick $\tau_j(\kappa)$ so that both $\tau_j(\kappa)\,\varepsilon_j(\kappa) = \mathbf{o}(\kappa^p)$ and $\tau_j(\kappa) \to \infty$ as $\kappa \to 0^+$. Further let $\mathbf{I}_{j,\kappa} = (\lambda_{j,\kappa} - \tau_j(\kappa)\,\varepsilon_j(\kappa), \lambda_{j,\kappa} + \tau_j(\kappa)\,\varepsilon_j(\kappa))$, and $\mathbf{I}_\kappa = \bigcup_{j=1}^m \mathbf{I}_{j,\kappa}$. Then $|\mathbf{I}_\kappa| = \mathbf{o}(\kappa^p)$ and, since $E_\kappa[\mathbf{I}_\kappa] = E_\kappa[\mathbf{I}_\kappa]\,P_0 + E_\kappa[\mathbf{I}_\kappa](1 - P_0)$, it follows from Lemma 2.2 that we only need prove that $E_\kappa[\mathbf{I}_\kappa]\,P_0 \to_s P_0$ as $\kappa \to 0^+$. The spectral theorem now gives

$$\begin{aligned} \varepsilon_j(\kappa)^2 &= \int_{-\infty}^{\infty} (\mu - \lambda_{j,\kappa})^2\, d\langle E_\kappa(\mu)\,\phi_{j,\kappa}, \phi_{j,\kappa}\rangle \\ &\geqslant \tau_j(\kappa)^2\,\varepsilon_j(\kappa)^2 \int_{\mathbf{J}_\kappa} d\langle E_\kappa(\mu)\,\phi_{j,\kappa}, \phi_{j,\kappa}\rangle \\ &= \tau_j(\kappa)^2\,\varepsilon_j(\kappa)^2\,\|(1 - E_\kappa[\mathbf{I}_{j,\kappa}])\,\phi_{j,\kappa}\|^2, \end{aligned}$$

where $\mathbf{J}_\kappa = \mathbf{R} \backslash \mathbf{I}_\kappa$.

Hence $\|(1 - E_\kappa[\mathbf{I}_{j,\kappa}])\,\phi_{j,\kappa}\| \leqslant 1/\tau_j(\kappa) \to 0$ as $\kappa \to 0^+$. Since $\|(1 - E_\kappa[\mathbf{I}_{j,\kappa}])(\phi_{j,\kappa} - \phi_j)\| \leqslant \|\phi_{j,\kappa} - \phi_j\| \to 0$ as $\kappa \to 0^+$, we have

$$\|(1 - E_\kappa[\mathbf{I}_\kappa])\,\phi_j\| \leqslant \|(1 - E_\kappa[\mathbf{I}_{j,\kappa}])\,\phi_j\| \to 0$$

as $\kappa \to 0^+$. Now $\phi_1, ..., \phi_m$ span $P_0\mathscr{H}$, so $E_\kappa[\mathbf{I}_\kappa]\,P_0$ converges uniformly to P_0 as $\kappa \to 0^+$, and (i) is proved.

To prove (ii), we simply note that if $H_0\phi = \lambda_0\phi$ with $\|\phi\| = 1$, then $(\lambda_\kappa, E_\kappa[\mathbf{I}_\kappa]\phi)$ is a p-pair for $\{H_\kappa\}$ near λ_0. This is a direct consequence of the spectral theorem since

$$\|(H_\kappa - \lambda_\kappa) E_\kappa[\mathbf{I}_\kappa]\phi\|^2 = \int_{\mathbf{I}_\kappa} (\mu - \lambda_\kappa)^2 \, d\langle E_\kappa(\mu)\phi, \phi\rangle = \mathbf{o}(\kappa^{2p}).$$

The following corollary is now obtained exactly as in the case when λ_0 is isolated [9].

COROLLARY 2.3. *The spectrum of* $\{H_\kappa\}$ *is concentrated near* λ_0 *whenever* $H_\kappa \to H_0$ *in the strong resolvent sense and* λ_0 *has finite multiplicity.*

This follows by letting $\lambda_{j,\kappa} = \lambda_0$ and $\phi_{j,\kappa} = (H_\kappa - i)^{-1}(H_0 - i)\phi_j = (\lambda_0 - i)(H_\kappa - i)^{-1}\phi_j$, where i is the imaginary unit. Then $\|(H_\kappa - \lambda_0)\phi_{j,\kappa}\| = \mathbf{o}(1)$ as $\kappa \to 0^+$.

To obtain more precise concentration estimates we invoke hypotheses that lead to analytic perturbation theory when λ_0 is isolated. Weaker hypotheses leading to asymptotic perturbation theory in the isolated case (cf. [6, 8]) would suffice for Theorem 2.4.

Assume now that

$$H_\kappa = H_0 + \kappa V,$$

with V symmetric and bounded relative to H_0. Then letting $\lambda_{j,\kappa} = \lambda_0$ and $\phi_{j,\kappa} = \phi_j$, $j = 1, ..., m$, yields

$$\|(H_\kappa - \lambda_{j,\kappa})\phi_{j,\kappa}\| = \mathcal{O}(\kappa),$$

so that the spectrum of $\{H_\kappa\}$ is concentrated near λ_0 to order p for any $p < 1$. Now for each $j = 1, ..., m$ let

$$\|(H_\kappa - \lambda_{j,\kappa})\phi_{j,\kappa}\| = \|(H_\kappa - \lambda_0)\phi_j\| = \varepsilon_j(\kappa),$$

and pick $\tau_j(\kappa)$ such that as $\kappa \to 0^+$, $\tau_j(\kappa) \to \infty$ and $\tau_j(\kappa)\,\varepsilon_j(\kappa) = \mathbf{o}(\kappa^p)$ for all $p < 1$. Denote by $\mathbf{I}_{j,\kappa}$ the interval $(\lambda_0 - \tau_j(\kappa)\,\varepsilon_j(\kappa), \lambda_0 + \tau_j(\kappa)\,\varepsilon_j(\kappa))$, and let $\mathbf{I}_\kappa = \bigcup_{j=1}^m \mathbf{I}_{j,\kappa}$. Then $E_\kappa[\mathbf{I}_\kappa] \to_s P_0$ as $\kappa \to 0^+$, and we are ready to formulate an embedded eigenvalue analogue of the first order concentration theorem, Theorem XII.23 of [8].

THEOREM 2.4. *If* λ_0 *has finite multiplicity and* $H_\kappa = H_0 + \kappa V$ *with* V *symmetric and bounded relative to* H_0, *the spectrum of* $\{H_\kappa\}$ *is concentrated to order* 1 *near* λ_0.

Proof. Choose the orthonormal basis $\phi_1, ..., \phi_m$ of $P_0\mathscr{H}$ as the eigenvectors of $P_0 V$ restricted to $P_0\mathscr{H}$. Then for $j = 1, ..., m$ let $\lambda_{j,\kappa} =$

$\lambda_0 + \kappa\langle V\phi_j, \phi_j\rangle$ and $\phi_{j,\kappa} = E_\kappa[\mathbf{I}_\kappa]\,\phi_j$, where $\mathbf{I}_\kappa$ was defined in the preceding paragraph. Then

$$\begin{aligned}\|(H_\kappa - \lambda_{j,\kappa})\,\phi_{j,\kappa}\| &= \|E_\kappa[\mathbf{I}_\kappa](H_0 + \kappa V - \lambda_0 - \kappa\langle V\phi_j, \phi_j\rangle)\,\phi_j\| \\ &= \kappa\,\|E_\kappa[\mathbf{I}_\kappa](V - \langle V\phi_j, \phi_j\rangle)\,\phi_j\| \\ &= \kappa\,\|(E_\kappa[\mathbf{I}_\kappa] - P_0)(V - \langle V\phi_j, \phi_j\rangle)\,\phi_j\|,\end{aligned}$$

since $P_0 V\phi_j = \langle V\phi_j, \phi_j\rangle\,\phi_j$. Now $E_\kappa[\mathbf{I}_\kappa] \to_s P_0$ as $\kappa \to 0^+$, so $\|(H_\kappa - \lambda_{j,\kappa})\,\phi_{j,\kappa}\| = \mathbf{o}(\kappa)$ for $j = 1, ..., m$, and the theorem is proved.

This shows that in the dilatation analytic case considered in [8, 10], the expansion coefficient a_1, is simply the first correction coefficient in a formal Rayleigh–Schrödinger series. To refine Theorem 2.4 we restrict our attention to simple eigenvalues, thus avoiding well-known technicalities associated with degenerate perturbation theory.

THEOREM 2.5. *Assume that λ_0 is simple with corresponding normalized eigenvector ϕ, and that $H_\kappa = H_0 + \kappa V$ with V symmetric and bounded relative to H_0. Further assume that $V\phi - \langle V\phi, \phi\rangle\phi \in \mathscr{R}(H_0 - \lambda_0)$, the range of $H_0 - \lambda_0$. Then the spectrum of $\{H_\kappa\}$ is concentrated to order 2 near λ_0.*

Proof. $H_0 - \lambda_0$ is one-to-one with dense range in the orthogonal complement of its null space, so we may define the (possibly unbounded) self adjoint operator S in $\mathscr{H}$ by $S\psi = (H_0 - \lambda_0)^{-1}\,\psi$ for $\psi \in \mathscr{R}(H_0 - \lambda_0)$, $S\phi = 0$, and linearity. As in the proof of Theorem 2.4 we employ formal Rayleigh–Schrödinger perturbation theory to define

$$\lambda_\kappa \equiv \lambda_{1,\kappa} = \lambda_0 + \kappa\langle V\phi, \phi\rangle - \kappa^2\langle SV\phi, V\phi\rangle,$$

and

$$\phi_\kappa \equiv \phi_{1,\kappa} = E_\kappa[\mathbf{I}_\kappa](\phi - \kappa SV\phi),$$

with $\mathbf{I}_\kappa$ as in the previous theorem. Then

$$\begin{aligned}&\|(H_\kappa - \lambda_\kappa)\,\phi_\kappa\| \\ &\quad = \kappa^2\,\|E_\kappa[\mathbf{I}_\kappa](-VSV\phi + \langle V\phi, \phi\rangle\,SV\phi + \langle SV\phi, V\phi\rangle\,\phi)\| + \mathscr{O}(\kappa^3) \\ &\quad = \kappa^2\,\|(E_\kappa[\mathbf{I}_\kappa] - P_0)(-VSV\phi + \langle V\phi, \phi\rangle\,SV\phi + \langle SV\phi, V\phi\rangle\,\phi)\| + \mathscr{O}(\kappa^3),\end{aligned}$$

since $P_0 VSV\phi = \langle SV\phi, V\phi\rangle\phi$ and $P_0 SV\phi = 0$. Thus since $E_\kappa[\mathbf{I}_\kappa \to_s P_0$ as $\kappa \to 0^+$, $\|(H_\kappa - \lambda_\kappa)\,\phi_\kappa\| = \mathbf{o}(\kappa^2)$ and the theorem follows.

Reference to [10] now shows that in the dilatation analytic case considered there, the condition $V\phi - \langle V\phi, \phi\rangle\phi \in \mathscr{R}(H_0 - \lambda_0)$ is sufficient to guarantee that the series coefficients a_2 and a_3 are real, and thus that the concentration is to order p for all $p < 4$.

3. Eigenvalues of Infinite Multiplicity

We now drop the assumption that λ_0 has finite multiplicity. It is remarked in [9] that when λ_0 has infinite multiplicity, strong resolvent convergence of H_κ to H_0 is not sufficient to guarantee spectral concentration near λ_0. Herein concentration results near any eigenvalue λ_0 of H_0 are obtained by restricting attention to the semibounded case, and employing techniques related to those used in [2, 4].

So let $\mathbf{h}_0(\cdot, \cdot)$ be a Hermitian symmetric bilinear form defined on a linear manifold $\mathscr{D}(\mathbf{h}_0)$ which is dense in $\mathscr{H}$. Further assume that the corresponding quadratic form is closed, and bounded below by the unit form,

$$\mathbf{h}_0(\chi) = \mathbf{h}_0(\chi, \chi) \geqslant \langle \chi, \chi \rangle, \; \chi \in \mathscr{D}(\mathbf{h}_0).$$

Since $\mathbf{h}_0(\cdot)$ is closed, $\mathscr{D}(\mathbf{h}_0)$ with the inner product

$$\langle \cdot, \cdot \rangle_0 = \mathbf{h}_0(\cdot, \cdot),$$

is a Hilbert space with norm $\|\cdot\|_0 = (\mathbf{h}_0(\cdot))^{1/2}$. To the form $\mathbf{h}_0(\cdot, \cdot)$ there corresponds a positive definite self adjoint operator H_0 in $\mathscr{H}$ which is defined on

$$\mathscr{D}(H_0) = \{\chi \in \mathscr{D}(\mathbf{h}_0) : \mathbf{h}_0(\chi, \cdot) \text{ is continuous on } \mathscr{D}(\mathbf{h}_0) \text{ in the topology of } \mathscr{H}\},$$

by

$$\langle H_0\chi, \psi \rangle = \mathbf{h}_0(\chi, \psi), \qquad \chi \in \mathscr{D}(\mathbf{h}_0), \quad H_0\chi \in \mathscr{H}, \quad \psi \in \mathscr{D}(\mathbf{h}_0),$$

(cf. [7]).

Now let $\mathbf{v}(\cdot, \cdot)$ be a Hermitian symmetric bilinear form defined on a linear manifold $\mathscr{D}(\mathbf{v})$ which is dense in $\mathscr{D}(\mathbf{h}_0)$ (and therefore in $\mathscr{H}$). Further let the corresponding quadratic form be nonnegative,

$$\mathbf{v}(\chi) \equiv \mathbf{v}(\chi, \chi) \geqslant 0, \qquad \chi \in \mathscr{D}(\mathbf{v}),$$

and closed in $\mathscr{D}(\mathbf{h}_0)$. It is easy to prove that $\mathbf{v}(\cdot)$ is closed in $\mathscr{D}(\mathbf{h}_0)$ if and only if $\mathbf{h}_0(\cdot) + \kappa\mathbf{v}(\cdot)$ is closed in $\mathscr{H}$ for some $\kappa > 0$ (and thus for all $\kappa > 0$). Denote by H_κ, $0 < \kappa \leqslant \kappa_0$, the positive definite self adjoint operator in $\mathscr{H}$ corresponding to $\mathbf{h}_0(\cdot, \cdot) + \kappa\mathbf{v}(\cdot, \cdot)$, i.e.,

$$\langle H_\kappa\chi, \psi \rangle = \mathbf{h}_0(\chi, \psi) + \kappa\mathbf{v}(\chi, \psi).$$

Then $H_\kappa \to H_0$ in the strong resolvent sense as $\kappa \to 0^+$.

The concentration theorem requires the introduction of another operator associated with this perturbation. Let $\mathscr{V}$ be the nonnegative self adjoint operator in $\mathscr{D}(\mathbf{h}_0)$ defined on

$$\mathscr{D}(\mathscr{V}) = \{\chi \in \mathscr{D}(\mathbf{v}) : \mathbf{v}(\chi, \cdot) \text{ is continuous on } \mathscr{D}(\mathbf{v}) \text{ in the topology of } \mathscr{D}(\mathbf{h}_0)\},$$

by

$$\mathbf{h}_0(\mathscr{V}\chi, \psi) = \mathbf{v}(\chi, \psi), \qquad \chi \in \mathscr{D}(\mathscr{V}), \quad \mathscr{V}\chi \in \mathscr{D}(\mathbf{h}_0), \quad \psi \in \mathscr{D}(\mathbf{v}).$$

Observe that $\mathscr{D}(H_\kappa) \subset \mathscr{D}(\mathscr{V})$, and that for $\chi \in \mathscr{D}(H_\kappa)$ and $\psi \in \mathscr{D}(\mathbf{v})$,

$$\mathbf{h}_0(\chi, \psi) + \kappa \mathbf{v}(\chi, \psi) = \langle H_\kappa \chi, \psi \rangle = \mathbf{h}_0((\kappa \mathscr{V} + 1)\, \chi, \psi),$$

and thus $H_0^{-1} H_\kappa \subset \kappa \mathscr{V} + 1$ (cf. [6, Corollary VI.2.4]).

THEOREM 3.1. *Under the hypotheses of this section, if there exists a Borel function* $F\colon [0, \infty) \to [1, \infty)$ *such that* $F(\nu) \to \infty$ *as* $\nu \to \infty$ *and* $\mathscr{D}(H_0) \subset \mathscr{D}(F(\mathscr{V}))$ *with* $\|F(\mathscr{V})\chi\|_0 \leqslant K \|H_0 \chi\|$, $K > 0$, $\chi \in \mathscr{D}(H_0)$, *the spectrum of* $\{H_\kappa\}$ *is concentrated near any eigenvalue* λ_0 *of* H_0. *When* $F(\nu) = (\nu + 1)^\gamma$, $0 < \gamma \leqslant \frac{1}{2}$, *the spectrum of* $\{H_\kappa\}$ *is concentrated to order* p *for all* $p < 2\gamma$ *near any eigenvalue* λ_0 *of* H_0.

Proof. By a lemma of [3], there exists $M > 0$ such that

$$\|\chi\| \leqslant M \, \|[F(\mathscr{V})]^{-1} \chi\|_0$$

for all $\chi \in \mathscr{D}(\mathbf{h}_0)$. Let P_0 be the orthogonal projection onto the λ_0-eigenspace, and let $\phi \in P_0 \mathscr{H} \subset \mathscr{D}(F(\mathscr{V}))$. Then since $H_0 \phi = \lambda_0 \phi$ and $H_\kappa^{-1} H_0 \subset (\kappa \mathscr{V} + 1)^{-1}$,

$$\begin{aligned}
\|(H_\kappa - \lambda_0)\, H_\kappa^{-1} \lambda_0 \phi\|^2 &= \lambda_0^2 \, \|[1 - (\kappa \mathscr{V} + 1)^{-1}]\, \phi\|^2 \\
&\leqslant M^2 \lambda_0^2 \, \|[F(\mathscr{V})]^{-1}\, [1 - (\kappa \mathscr{V} + 1)^{-1}]\, \phi\|_0^2 \\
&= M^2 \lambda_0^2 \int_0^\infty F(\mu)^{-2}\, [1 - (\kappa\mu + 1)^{-1}]^2 \, d\langle E(\mu)\phi, \phi\rangle_0 \\
&\leqslant M^2 \lambda_0^2 \sup_{\nu \in (0,\infty)} \left(\frac{\kappa^2 \nu^2}{(\kappa\nu + 1)^2} F(\nu)^{-4} \right) \\
&\qquad \times \int_0^\infty F(\mu)^2 \, d\langle E(\mu)\phi, \phi\rangle_0 \\
&= M^2 \lambda_0^2 \sup_{\nu \in (0,\infty)} \left(\frac{\kappa^2 \nu^2}{(\kappa\nu + 1)^2} F(\nu)^{-4} \right) \cdot \|F(\mathscr{V})\phi\|_0^2 \\
&\leqslant M^2 K^2 \lambda_0^4 \sup_{\nu \in (0,\infty)} \left(\frac{\kappa^2 \nu^2}{(\kappa\nu + 1)^2} F(\nu)^{-4} \right) \\
&\equiv \varepsilon(\kappa) = \mathbf{o}(1),
\end{aligned} \tag{3.1}$$

since $F(\nu) \to \infty$ as $\nu \to \infty$. Herein $E(\cdot)$ denotes the resolution of the identity for the self adjoint operator $\mathscr{V}$.

Now pick $\tau(\kappa)$ such that $\tau(\kappa) \to \infty$ and $\tau(\kappa)\,\varepsilon(\kappa) \to 0$ as $\kappa \to 0^+$. Let $\mathbf{I}_\kappa = (\lambda_0 - \tau(\kappa)\,\varepsilon(\kappa), \lambda_0 + \tau(\kappa)\,\varepsilon(\kappa))$, and for $\phi \in P_0\mathscr{H}$ with $\|\phi\| = 1$, let $\lambda_\kappa = \lambda_\kappa(\phi) = \lambda_0$ and $\phi_\kappa = \lambda_0 H_\kappa^{-1}\phi$. Then if $E_\kappa(\cdot)$ is the resolution of the identity for H_κ, (3.1) gives

$$\begin{aligned}\varepsilon(\kappa)^2 &\geqslant \|(H_\kappa - \lambda_0)\,\phi_\kappa\|^2 = \int_0^\infty (\mu - \lambda_0)^2\, d\langle E_\kappa(\mu)\,\phi_\kappa, \phi_\kappa\rangle \\ &\geqslant \tau(\kappa)^2\,\varepsilon(\kappa)^2 \int_{\mathbf{J}_\kappa} d\langle E_\kappa(\mu)\,\phi_\kappa, \phi_\kappa\rangle = \tau(\kappa)^2\,\varepsilon(\kappa)^2\,\|(1 - E_\kappa[\mathbf{I}_\kappa])\,\phi_\kappa\|^2,\end{aligned}$$

where $\mathbf{J}_\kappa = [0, \infty)\backslash\mathbf{I}_\kappa$. Thus $\|(1 - E_\kappa[\mathbf{I}_\kappa])\,\phi_\kappa\|^2 \leqslant 1/\tau(\kappa) \to 0$ as $\kappa \to 0^+$. Since $\|(1 - E_\kappa[\mathbf{I}_\kappa])(\phi_\kappa - \phi)\| \leqslant \|\phi_\kappa - \phi\| \to 0$ as $\kappa \to 0^+$, we have

$$(1 - E_\kappa[\mathbf{I}_\kappa])\,P_0 \to_s 0 \qquad \text{as} \quad \kappa \to 0^+.$$

Since Lemma 2.2 implies that $E_\kappa[\mathbf{I}_\kappa](1 - P_0) \to_s 0$ as $\kappa \to 0^+$, it follows that $E_\kappa[\mathbf{I}_\kappa] \to_s P_0$ as $\kappa \to 0^+$; i.e., the spectrum of $\{H_\kappa\}$ is concentrated near λ_0. When $F(v) = (v+1)^\gamma$, $0 < \gamma \leqslant \frac{1}{2}$, the supremum in (3.1) is $\mathcal{O}(\kappa^{4\gamma})$ as $\kappa \to 0^+$ and the theorem is proved.

A simple consequence of Theorem 3.1 is that if $\mathbf{v}(\cdot)$ is relatively bounded with respect to $\mathbf{h}_0(\cdot)$, then the spectrum of $\{H_\kappa\}$ is concentrated to order p for any $p < 1$ near any eigenvalue of H_0. In this case an obvious choice for F is $F(v) = v + 1$.

References

1. C. C. Conley and P. A. Rejto, Spectral concentration II – general theory, *in* "Perturbation Theory and its Applications in Quantum Mechanics" (C. H. Wilcox, Ed.), pp. 129–143, Wiley, New York, 1966.
2. W. M. Greenlee, On spectral concentration for semi-bounded operators, *J. Funct. Anal.* **5** (1970), 66–70.
3. W. M. Greenlee, Singular perturbation of simple eigenvalues, *Rocky Mountain J. Math.* **6** (1976), 575–584.
4. W. M. Greenlee, Stability theorems for singular perturbation of eigenvalues, *Manuscripta Math.* **34** (1981), 157–174.
5. J. S. Howland, Puiseux series for resonances at an embedded eigenvalue, *Pacific J. Math.* **55** (1974), 157–176.
6. T. Kato, "Perturbation Theory for Linear Operators," Springer-Verlag, Berlin/Heidelberg/New York, 1966.
7. J. L. Lions, Equations Différentielles Opérationelles et Problèmes aux Limites," Springer-Verlag, Berlin/Heidelberg/New York, 1961.
8. M. Reed and B. Simon, "Methods of Modern Mathematical Physics IV: Analysis of Operators," Academic Press, New York/London, 1978.
9. R. C. Riddell, Spectral concentration for self-adjoint operators, *Pacific J. Math.* **23** (1967), 377–401.
10. B. Simon, Resonances in n-body quantum systems with dilatation analytic potentials and the foundations of time-dependent perturbation theory, *Ann. of Math.* **97** (1973), 247–274.